Volterra-Type Integral Equations of the Second Kind with Nonsmooth Solutions: High-Order Methods Based on Collocation Techniques

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Methods with arbitrary orders of convergence are derived for the approximate solution of second-kind Volterra integral equations with weakly singular kernels where exact solutions have unbounded derivatives at the left endpoint of the interval of integration. These methods are based on collocation techniques in certain nonsmooth piecewise function spaces whose elements reflect the singular behavior of the given equation.

1. Introduction

Many modeling problems in mathematical physics (compare, for example, te Riele [9]) lead to integral equations of the second kind with weakly singular kernels, i.e.,

$$y(t)=g(t)+\int_0^t \frac{G(t,s)}{(t-s)^\alpha}\gamma(s,y(s))\,ds, \qquad t\in I:=[0,T], \tag{1.1}$$

where $\alpha=\frac{1}{2}$. Here, $G\in C(S)$ [with $S:=\{(t,s):0\leqslant s\leqslant t\leqslant T\}$], γ is continuous with respect to s and (uniformly) Lipschitz continuous with respect to y [we note that γ is often given by $\gamma(s,y)=\text{const.}\,y^r$, $r\in\mathbb{N}$], and g is assumed to have the form

$$g(t)=g_1(t)+t^\beta g_2(t), \qquad \beta=1-\alpha \tag{1.2}$$

with g_1, g_2 smooth on I.

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It has been shown [8, 4, 7] that the (unique) solution y of (1.1) is continuous on I but has unbounded derivatives at $t = 0$ [i.e., $y'(t) = O(t^{\beta-1})$ for $t \to 0_+$]; for additional details see also Section 2 of this paper. The fact that the kernel function $K(t, s, y(s)) := G(t, s)\gamma(s, y(s))$ is not smooth near $t = 0$ will have an influence on the accuracy of a numerical method near the beginning of the interval of integration; this problem has been studied by Linz [6] and de Hoog and Weiss [5] (see also Miller and Feldstein [7]). While those methods are based on piecewise polynomial interpolation and on appropriate product integration techniques, te Riele [9] uses nonpolynomial collocation techniques (together with suitable weighted quadrature formulas for the discretization of the resulting moment integrals) to simulate the behavior of the exact solution near the origin. However, the obtained rate of convergence was unsatisfactory [9, Theorem 3.1].

The present paper may be regarded as a sequel to a previous paper [9]. Here, we investigate projection methods with respect to nonsmooth approximation spaces which differ somewhat from the spaces used there [9]; this choice allows for the construction of collocation methods of *arbitrary* order. For general equations of the form (1.1) (i.e., $\partial G/\partial t \not\equiv 0$) these methods will usually only be applied to generate a certain number of sufficiently accurate starting values to be used later in a more efficient (finite-difference) method [note that, if $\partial G/\partial t \not\equiv 0$, then a change of t in (1.1) will necessitate a new evaluation of the entire integral term]. However, most physical applications lead to equations where $G(t, s) \equiv$ const. for $(t, s) \in S$ (compare the extensive list of examples of te Riele [9]); hence, for these special integral equations the collocation methods described in the following sections become much more competitive if used on the entire interval I.

The outline of the paper is as follows. In Section 2 we describe the smoothness properties of the exact solution y of (1.1) and introduce the approximation spaces in which this solution will be approximated by collocation. Sections 3 and 4 deal with the convergence analysis of the method, with a number of special cases, and with suggestions for possible implementations of the method. In Section 5 we touch briefly upon the connection of these methods with certain Runge–Kutta-type methods for (1.1) and some related open problems. Finally, a number of numerical illustrations are presented in Section 6.

2. Preliminaries

In the following we assume that $\alpha = \frac{1}{2}$. Suppose in addition that the forcing function g in (1.1) has the form (1.2), with $g_i \in C^{m+1}(I)$ $(i = 1,2)$ for some $m \in \mathbb{N} \cup \{0\}$. Like de Hoog and Weiss [4, 5], we associate with (1.1) the system of integral equations given by

$$v(t)=g_1(t)+\int_0^t \frac{s^\alpha}{(t-s)^\alpha} G(t,s)\gamma_1(s,v(s),w(s))\,ds, \qquad (2.1a)$$

$$w(t)=g_2(t)+t^{-\alpha}\int_0^t \frac{1}{(t-s)^\alpha} G(t,s)\gamma_2(s,v(s),w(s))\,ds, \qquad (2.1b)$$

where the kernel functions γ_i are defined by

$$\gamma_1(s,v,w) := \frac{1}{2s^\alpha}[\gamma(s,v+s^\alpha w)-\gamma(s,v-s^\alpha w)], \qquad (2.2a)$$

$$\gamma_2(s,v,w) := \tfrac{1}{2}[\gamma(s,v+s^\alpha w)+\gamma(s,v-s^\alpha w)]. \qquad (2.2b)$$

Observe that if $\gamma(s,y)=y$ (linear integral equation), then γ_1 and γ_2 are independent of v and w, respectively, and the corresponding system (2.1) reduces to

$$v(t)=g_1(t)+\int_0^t \frac{s^\alpha G(t,s)}{(t-s)^\alpha} w(s)\,ds, \qquad (2.3a)$$

$$w(t)=g_2(t)+t^{-\alpha}\int_0^t \frac{G(t,s)}{(t-s)^\alpha} v(s)\,ds. \qquad (2.3b)$$

It follows that, under the hypotheses mentioned and with

i. $G\in C^{m+1}(S)$,
ii. $(\partial^{m+1}/\partial s^{m+1})\gamma(s,y)\in C(I\times R')$, and
iii. $\partial^{2m+2}/\partial y^{2m+2}\in C(I\times R')$

(where $R':=\{z:|y(t)-z|<\rho,\ t\in I\}$), the exact solution y of (1.1) may be expressed

$$y(t)=v(t)+t^\beta w(t), \qquad t\in I, \qquad (2.4)$$

where $v,w\in C^{m+1}(I)$. [However, it has already been observed [5] that the boundedness of the solution y of (1.1) does, in general, not imply that v and w are bounded (2.1). Hence, a stable equation (1.1) may give rise to an unstable system (2.1): A simple example is given by $\alpha=\frac{1}{2}$, $g_1(t)\equiv 1$, $g_2(t)\equiv 0$, $G(t,s)\equiv -1$, $\gamma(s,y)=y$; here $y(t)\to 0$, $t\to\infty$, while $v(t)=\exp(\pi t)$, $|w(t)|=O(\exp(\pi t)/\sqrt{t})$, $t\to\infty$.] Note, furthermore, that the presence of the factor $t^{-\alpha}$ in (2.1b) [or in (2.3b)] implies that v and w satisfy the above smoothness result only if $\alpha=\frac{1}{2}$, if (for example) $\alpha\in(\frac{1}{2},1)$; otherwise integrals of the form

$$t^{-\alpha}\int_0^t (t-s)^{-\alpha}\,ds = t^{1-2\alpha}/(1-\alpha),$$

for example, no longer remain bounded on $[0,T]$ $(T>0)$.

If (2.4) holds, then for $t \in \sigma_n$ [where $\sigma_0 := [0, t_1]$, $\sigma_n := (t_n, t_{n+1}]$, $t_n := nh$, with $t_N = T$] we write y as

$$y(t) = \sum_{j=0}^{m_1} v_{n,j}\tau^j + h^\beta(n+\tau)^\beta \sum_{j=0}^{m_2} w_{n,j}\tau^j + R_n^{\mathrm{I}}(\tau) + h^\beta(n+\tau)^\beta R_n^{\mathrm{II}}(\tau), \tag{2.5}$$

with

$$v_{n,j} := h^j v^{(j)}(t_n)/j!,$$
$$w_{n,j} := h^j w^{(j)}(t_n)/j!,$$
$$\tau = (t - t_n)/h,$$

and

$$R_n^{\mathrm{I}}(\tau) := h^{m_1+1}\tau^{m_1+1}v^{(m_1+1)}(\xi_n)/(m_1+1)! \qquad (t_n < \xi_n < t_n + \tau h),$$
$$R_n^{\mathrm{II}}(\tau) := h^{m_2+1}\tau^{m_2+1}w^{(m_2+1)}(\eta_n)/(m_2+1)! \qquad (t_n < \eta_n < t_n + \tau h). \tag{2.6}$$

Here, m_1 and m_2 are suitable nonnegative integers satisfying $m_1 \leqslant m$, $m_2 \leqslant m$; their choice will be made more precise below. Note that for $\beta = 1$ and $m_1 = m_2 = m$ (2.5) reduces to the classical Taylor formula representation for y on σ_n.

The representation (2.5) for the exact solution y of (1.1) [subject to (2.4)] yields the motivation for the choice of the projecting space in which an approximate solution u will be sought: If y is such that v and w in (2.4) satisfy $v \in \Pi_{m_1}$, $w \in \Pi_{m_2}$ [i.e., $R_n^{\mathrm{I}}(\tau) \equiv 0$, $R_n^{\mathrm{II}}(\tau) \equiv 0$ for all $n = 0, \ldots, N-1$], then y will be required to be an element (namely, u) of this space. Hence, we define, setting $Z_N := \{t_n = nh : n = 0, \ldots, N \ (t_N = T > 0)\}$,

$$V_m^{(-1)}(Z_n) := \left\{ u : u|_{\sigma_n} = u_n(t) := p_n(\tau) + h^\beta(n+\tau)^\beta q_n(\tau),\ n = 0, \ldots, N-1 \right\} \tag{2.7}$$

with

$$p_n(\tau) := \sum_{j=0}^{m_1} \alpha_{n,j}\tau^j, \quad q_n(\tau) := \sum_{j=0}^{m_2} \beta_{n,j}\tau^j \qquad [\tau = (t - t_n)/h].$$

If $\beta = 1$ and $m_1 = m_2 = m$, then $V_m^{(-1)}(Z_N)$ coincides with $S_m^{(-1)}(Z_N)$, the space of piecewise polynomials of degree m that possess (finite) discontinuities at their knots Z_N [cf. 3]. We have

$$\dim V_m^{(-1)}(Z_N) = N(m_1 + m_2 + 2) \qquad (0 < \beta < 1), \tag{2.8}$$

and

$$\dim S_m^{(-1)}(Z_N) = N(m+1) \qquad (\beta = 1, \quad m_1 = m_2 = m). \tag{2.9}$$

3. Collocation in $V_m^{(-1)}(Z_N)$

For the subsequent analysis we define $\mu := m_1 + m_2 + 1$ (if $0 < \beta < 1$; for $\beta = 1$ and $m_1 = m_2 = m$ we set $\mu = m$), and we introduce the sets $X(N) := \bigcup_{n=0}^{N-1} X_n$, where

$$X_N := \{t_n + c_i h : 0 \leqslant c_0 < \cdots < c_\mu = 1\}. \tag{3.1}$$

Consider now (1.1) for $t \in \sigma_n$. We have

$$y(t) = \int_{t_n}^{t} (t-s)^{-\alpha} G(t,s) \gamma(s, y(s))\, ds + F_n(t), \tag{3.2}$$

where

$$F_n(t) := g(t) + \int_{t_0}^{t_n} (t-s)^{-\alpha} G(t,s) \gamma(s, y(s))\, ds \qquad (n = 0, \ldots, N-1). \tag{3.3}$$

An element $u \in V_m^{(-1)}(Z_N)$ approximating the exact solution y of (1.1) on I is then determined from the recursion

$$\begin{aligned} u_n(t_n + c_i h) = {} & h^{1-\alpha} \int_0^{c_i} (c_i - \tau)^{-\alpha} G(t_n + c_i h, t_n + \tau h) \\ & \times \gamma(t_n + \tau h, u_n(t_n + \tau h))\, d\tau \\ & + \hat{F}_n(t_n + c_i h) \qquad (i = 0, \ldots, \mu,\ n = 0, \ldots, N-1), \end{aligned} \tag{3.4}$$

with

$$\begin{aligned} \hat{F}_n(t) := {} & g(t) + h \sum_{k=0}^{n-1} \int_0^1 (t - t_k - \tau h)^{-\alpha} G(t, t_k + \tau h) \\ & \times \gamma(t_k + \tau h, u_k(t_k + \tau h))\, d\tau \end{aligned} \tag{3.5}$$

approximating the "history" (or lag) term (3.3). Note that, for $c_0 > 0$, the integrands in (3.5) are no longer singular; also, (3.4) does not, of course, require knowledge of any starting values.

By using the standard contraction mapping arguments, together with the hypotheses governing the existence of a unique solution for (1.1) on I, one can show that (3.4) has a unique solution $u \in V_m^{(-1)}(Z_N)$ whenever $h > 0$ is sufficiently small. [For remarks on the numerical evaluation of the integrals in (3.4) and (3.5) compare te Riele [9].]

Let now $e(t) := y(t) - u(t)$, with $e|_{\sigma_n} := e_n(t)$. It follows from (3.2) and (3.4) and from the hypotheses for $\gamma(s, y)$ that $e(t)$ satisfies

$$e_n(t_n + c_i h) = h^{1-\alpha}\int_0^{c_i}(c_i - \tau)^{-\alpha}G(t_n + c_i h, t_n + \tau h) \times \frac{\partial\gamma}{\partial y}(t_n + \tau h, \cdot)e_n(t_n + \tau h)\,d\tau + \Phi_n(t_n + c_i h) \qquad (i = 0,\ldots,\mu,\ n = 0,\ldots,N-1); \quad (3.6)$$

here,

$$\Phi_n(t_n + c_i h) = F_n(t_n + c_i h) - \hat{F}_n(t_n + c_i h) = h^{1-\alpha}\sum_{k=0}^{n-1}\int_0^1(n + c_i - k - \tau)^{-\alpha}G(t_n + c_i h, t_k + \tau h) \times \frac{\partial\gamma}{\partial y}(t_k + \tau h, \cdot)e_k(t_k + \tau h)\,d\tau. \quad (3.7)$$

Recall now the definition of $V_m^{(-1)}(Z_N)$: If we define the numbers $\{\delta_{n,j}\}$ and $\{\varepsilon_{n,j}\}$ by [see (2.5)]

$$h^{m_1+1}\delta_{n,j} := v_{n,j} - \alpha_{n,j} \qquad (j = 0,\ldots,m_1), \quad (3.8a)$$

$$h^{m_2+1}\varepsilon_{n,j} := w_{n,j} - \beta_{n,j} \qquad (j = 0,\ldots,m_2), \quad (3.8b)$$

then we may write

$$e_n(t_n + \tau h) = h^{m_1+1}\left\{\sum_{j=0}^{m_1}\delta_{n,j}\tau^j + \tau^{m_1+1}v^{(m_1+1)}(\xi_n)/(m_1+1)!\right\} + h^{m_2+\beta+1}(n+\tau)^\beta \times \left\{\sum_{j=0}^{m_2}\varepsilon_{n,j}\tau^j + \tau^{m_2+1}w^{(m_2+1)}(\eta_n)/(m_2+1)!\right\}$$
$$(n = 0,\ldots,N-1;\ 0 \leqslant m_1 \leqslant m,\ 0 \leqslant m_2 \leqslant m). \quad (3.9)$$

Observe that, for all $n = 0,\ldots,N-1$ $(N \geqslant 1)$, we have $h^\beta(n+\tau)^\beta \leqslant [h(n+1)]^\beta \leqslant (Nh)^\beta = (T - t_0)^\beta < \infty$. Substitution of (3.9) in (3.6) and (3.7) yields a (linear) system (in $\mathbb{R}^{\mu+1}$) for the vectors $\{(\delta_{n,0},\ldots,\delta_{n,m_1};\ \varepsilon_{n,0},\ldots,\varepsilon_{n,m_2})\}$, and available techniques [2, 3, 9] can then be used to derive the following result.

Theorem 1. *Let* $\alpha = \frac{1}{2}$, *and suppose that* g_1, g_2, G, *and* γ *in* (1.1) *and* (1.2) *satisfy the hypotheses stated following* (2.3). *If* $u \in V_m^{(-1)}(Z_N)$ *is determined* *from* (3.4), *then, for* $h \to 0_+$, $Nh = T - t_0$,

$$|e(t)| \leqslant C_1 h^{m_1+1} + C_2 (T - t_0)^{\beta} h^{m_2+1}, \qquad t \in I$$
$$\left[= O(h^{q+1}),\, q := \min(m_1, m_2) \right]. \tag{3.10}$$

We list a number of special cases illustrating various possible choices for m_1 and m_2.

a. $m_1 = m$, $m_2 = 0$ [$q = \min(m_1, m_2) = 0$]: The space $V_m^{(-1)}(Z_N)$ is generated by the $m+2$ functions $\{1, \tau, \ldots, \tau^m;\ (n+\tau)^{\beta}\}$; we have $\dim V_m^{(-1)}(Z_N) = N(m+2)$ and $\mu = m+1$. Thus Theorem 1 yields

$$|e(t)| = O(h) \qquad (h \to 0_+, Nh = T - t_0).$$

b. $m_1 = 0$, $m_2 = m$ ($q = 0$): Here, $V_m^{(-1)}(Z_N)$ is generated by the $m+2$ functions $\{1; (n+\tau)^{\beta}, (n+\tau)^{\beta}\tau, \ldots, (n+\tau)^{\beta}\tau^m\}$; we have $\dim V_m^{(-1)}(Z_N) = N(m+2)$, and $\mu = m+1$. Theorem 1 yields

$$|e(t)| = O(h) \qquad (h \to 0_+, Nh = T - t_0).$$

c. Let m be *odd*; $m_1 = m_2 = (m-1)/2$ [$q = (m-1)/2$; $\mu = m$]: $V_m^{(-1)}(Z_N)$ is generated by the $m+1$ functions $\{1, \tau, \ldots, \tau^{(m-1)/2};\ (n+\tau)^{\beta}, (n+\tau)^{\beta}\tau, \ldots, (n+\tau)^{\beta}\tau^{(m-1)/2}\}$; we have $\dim V_m^{(-1)}(Z_N) = N(m+1)$, and

$$|e(t)| = O(h^{(m+1)/2}).$$

d. Let m be *even*; $m_1 = m/2$, $m_2 = m/2 - 1$ ($q = m/2 - 1$; $\mu = m$): $V_m^{(-1)}(Z_N)$ is generated by $m+1$ functions $\{1, \tau, \ldots, \tau^{m/2};\ (n+\tau)^{\beta}, \ldots, (n+\tau)^{\beta}\tau^{(m/2)-1}\}$; we have $\dim V_m^{(-1)}(Z_N) = N(m+1)$. From Theorem 1,

$$|e(t)| = O(h^{m/2}).$$

We have already briefly mentioned the case where $\alpha = 0$ [i.e., (1.1) with regular kernel and smooth solution $y \in C^{m+1}(I)$]: If we select $m_1 = m_2 = m$ in (2.7) ($\alpha = 0$), then $V_m^{(-1)}(Z_N)$ degenerates to $S_m^{(-1)}(Z_N)$, with $\dim S_m^{(-1)}(Z_N) = N(m+1)$, and thus (3.10) reduces to the well-known convergence result $|e(t)| \leqslant Ch^{m+1}$, $t \in I$ (compare [3]). More generally, the "balanced" choice $m_1 = m_2$ yields, for $\alpha = \frac{1}{2}$, $|e(t)| \leqslant \tilde{C} h^{m_1+1}$, $t \in I$. Furthermore, if we are given the constraint $m_1 + m_2 = m - 1$ ($m \geqslant 1$) [i.e., if $V_m^{(-1)}(Z_N)$ is to be generated by precisely $m+1$ functions], then (3.10) leads to the optimal order of convergence if m_1 and m_2 are chosen as in (c) and (d), respectively.

In previous work [9] a somewhat different strategy was used for choosing the approximating space: Instead of $V_m^{(-1)}(Z_N)$, the author considered (for the case $\alpha = \frac{1}{2}$)

$$W_m^{(-1)}(Z_N) := \begin{cases} u: u|_{\sigma_0} = u_0(t) = \sum_{j=0}^{m} a_{0,j}\tau^{j/2}, & \tau = (t - t_0)/h, \\ u: u|_{\sigma_n} = u_n(t) = \sum_{j=0}^{m} a_{n,j}\tau^{j}, & \tau = (t - t_n)/h, \\ & 0 < n \leqslant N-1, \end{cases} \tag{3.11}$$

i.e., on subintervals other than the initial interval σ_0, piecewise polynomials of degree m are used, while on σ_0 itself the choice of the basis functions reflects the singular behavior of $y^{(j)}(t_0)$ $(j > 0)$ [recall (2.4)]. However, as indicated by Theorem 3.1 of te Riele [9], there seems to be a marked drop in the resulting order of convergence (but compare the remark made at the end of this section).

It is clear that, ideally, one would like to attain the same order of convergence (i.e., $p = m + 1$) independent of whether $\alpha = 0$ or $\alpha = \frac{1}{2}$ (recall that this is possible for the projecting space $S_m^{(-1)}(Z_N)$ provided the solution y of (1.1) satisfies $y \in C^{m+1}(I)$ [2, 3]), without a significant increase of the computational effort. A glimpse at (3.4) and (3.5) reveals that it is the evaluation of the "history term" (or "lag term") $\hat{F}_n(t)$ at $t = t_n + c_i h$ $(i = 0,\dots,\mu)$ that, as n becomes large, makes up the overwhelming part of the computational cost. Therefore, by (3.10), (3.4), and the fact that $|y^{(j)}(t_n)| \leqslant M < \infty, j = 0,\dots,m+1$, if $t_n \in [\varepsilon, T]$, $\varepsilon > 0$, the following modification of the collocation scheme analyzed above seems suggestive (see also Section 4):

i. On σ_n $(0 \leqslant n \leqslant r-1$, where the choice of $r \geqslant 1$ will depend on the given stepsize h), let $X_n := \{t_n + c_i h : 0 \leqslant c_0 < \cdots < c_{\mu_0} = 1\}$ with $\mu_0 := 2m+1$ $(m \geqslant 0)$, and set

$$u_n(t) := \sum_{j=0}^{m} \alpha_{n,j}\tau^j + h^{\beta} \sum_{j=0}^{m} \beta_{n,j}\tau^j(n+\tau)^{\beta} \qquad [\tau = (t - t_n)/h]. \tag{3.12a}$$

ii. If $n \geqslant r$ $(n \leqslant N-1)$, then let $X_n := \{t_n + \tilde{c}_i h : 0 \leqslant \tilde{c}_0 < \cdots < \tilde{c}_{\mu_1} = 1\}$, where now $\mu_1 := m$, the corresponding approximating functions will have the form

$$u_n(t) := \sum_{j=0}^{m} \tilde{a}_{n,j}\tau^j \qquad [\tau = (t - t_n)/h]. \tag{3.12b}$$

Hence, on $\sigma_0,\dots,\sigma_{r-1}$ $(t_r \geqslant \varepsilon > 0$ for a given $\varepsilon)$, and $\sigma_r,\dots,\sigma_{N-1}$ the approximations (3.12a) and (3.12b) will be determined, respectively, by (3.4)

($n = 0,\ldots,r-1$; μ replaced by μ_0) and by (3.4) ($n = r,\ldots,N-1$, μ replaced by μ_1).

We observe that method A of [9] is related to the scheme outlined above. It selects $r = 1$ (independent of the stepsize h) and, on σ_0, replaces the approximation given in (3.12a) by

$$\tilde{u}_0(t) := \sum_{j=0}^{[m/2]} \tilde{\alpha}_{0,j}\tau^j + h^\beta \sum_{j=0}^{[(m-1)/2]} \tilde{\beta}_{0,j}\tau^{j+\beta} \qquad (\text{with } \beta = \tfrac{1}{2}).$$

This choice corresponds to $m_1 = [m/2]$, $m_2 = [(m-1)/2]$ [with $m_1 + m_2 = m-1$ ($m \geqslant 1$)] in examples (c) and (d) following Theorem 1.

In order to analyze the resulting order of convergence ($h \to 0_+$, $Nh = T - t_0$ fixed) we first note that the representation (3.9) [which involves the hypotheses $v \in C^{m+1}(I)$, $w \in C^{m+1}(I)$] now becomes, respectively,

$$e_n(t_n + \tau h) = h^{m+1}\left\{ \sum_{j=0}^{m} \delta_{n,j}\tau^j + \tau^{m+1}v^{(m+1)}(\xi_n)/(m+1)! \right.$$
$$\left. + h^\beta (n+\tau)^\beta \left(\sum_{j=0}^{m} \varepsilon_{n,j}\tau^j + \tau^{m+1}w^{(m+1)}(\eta_n)/(m+1)! \right) \right\}$$
$$\left[\tau = (t - t_n)/h,\ t \in \sigma_n,\ n = 0(1)r-1\right] \qquad (3.13\text{a})$$

and

$$e_n(t_n + \tau h) = h^{m+1}\left\{ \sum_{j=0}^{m} \tilde{\delta}_{n,j}\tau^j + \tau^{m+1}y^{(m+1)}(z_n)/(m+1)! \right\}$$
$$\left[\tau = (t - t_n)/h,\ t \in \sigma_n,\ n = r,\ldots,N-1\right]. \qquad (3.13\text{b})$$

In (3.13a) r is such that, for a given $\varepsilon > 0$, $t_r \geqslant \varepsilon$; in (3.13b) we have set $h^{m+1}\tilde{\delta}_{n,j} := h^j y^{(j)}(t_n)/j! - \tilde{\alpha}_{n,j}$ ($j = 0,\ldots,m$; $n \geqslant r$). This suggests that the following result will hold.

Theorem 2. *Suppose that the assumptions of Theorem* 1 *hold, and let* $\varepsilon > 0$ *be given. If the projection* u *of* y *has the form* (3.12) *and satisfies* (1.1) *on the sets of collocation points indicated in* (3.12), *then the resulting error function* $e := y - u$ *satisfies, for* $t \in I$,

$$|e(t)| \leqslant Ch^{m+1} \qquad (h \to 0_+,\ Nh = T - t_0). \qquad (3.14)$$

Since the proof of the above result proceeds essentially along the lines of the one for Theorem 1 [cf. 3], we omit most of the details. However, in order to exhibit the necessary modifications we shall discuss the structure of the error equation. First, the collocation equation corresponding to (3.12a) and

(3.12b) may be written as $(n \geqslant r)$

$$
\begin{aligned}
& u_n(t_n + c_i h) \\
& \quad = h^{1-\alpha} \int_0^{c_i} (c_i - \tau)^{-\alpha} G(t_n + c_i h, t_n + \tau h) \gamma(t_n + \tau h, u_n(t_n + \tau h)) \, d\tau \\
& \quad + h^{1-\alpha} \sum_{k=r}^{n-1} \int_0^1 (n + c_i - k - \tau)^{-\alpha} G(t_n + c_i h, t_k + \tau h) \\
& \quad \times \gamma(t_k + \tau h, u_k(t_k + \tau h)) \, d\tau + \hat{\Psi}_r(t_n + c_i h) \qquad (i = 0, \ldots, \mu_1),
\end{aligned} \tag{3.15}
$$

with

$$
\begin{aligned}
\hat{\Psi}_r(t) := g(t) + h \sum_{k=0}^{r-1} \int_0^1 (t - t_k - \tau h)^{-\alpha} \\
\times G(t, t_k + \tau h) \gamma(t_k + \tau h, u_k(t_k + \tau h)) \, d\tau
\end{aligned} \tag{3.16}
$$

[observe the analogy with (3.5)], discretizing the given equation (1.1), written as $(t \geqslant t_n,\ n \geqslant r)$

$$
y(t) = \int_{t_r}^{t} (t - s)^{-\alpha} G(t, s) \gamma(s, y(s)) \, ds + \Psi_r(t), \tag{3.17}
$$

where

$$
\Psi_r(t) := g(t) + \int_{t_0}^{t_r} (t - s)^{-\alpha} G(t, s) \gamma(s, y(s)) \, ds. \tag{3.18}
$$

[Note that, for $t \geqslant t_r$ $(\geqslant \varepsilon > 0)$, the exact solution of (3.17) satisfies $y \in C^{m+1}(I)$.] Accordingly, the error function satisfies the recursion [letting $K_{n,i}(t) := G(t_n + c_i h, t)(\partial\gamma/\partial y)(t, \cdot)$]

$$
\begin{aligned}
e_n(t_n + c_i h) &= h^{1-\alpha} \int_0^{c_i} (c_i - \tau)^{-\alpha} K_{n,i}(t_n + \tau h) e_n(t_n + \tau h) \, d\tau \\
&\quad + h^{1-\alpha} \sum_{k=r}^{n-1} \int_0^1 (n + c_i - k - \tau)^{-\alpha} K_{n,i}(t_k + \tau h) e_k(t_k + \tau h) \, d\tau \\
&\quad + \{\Psi_r(t_n + c_i h) - \hat{\Psi}_r(t_n + c_i h)\} \\
&\qquad (i = 0, \ldots, \mu_1;\ n = r, \ldots, N-1).
\end{aligned} \tag{3.19}
$$

On the (fixed) subinterval $I_\varepsilon := [t_0, t_0 + \varepsilon]$ (recall that, for a given $\varepsilon > 0$, r is such that $t_r \geqslant \varepsilon$ for $h > 0$, and hence for $h \to 0_+$) we may invoke Theorem 1: (3.10) yields, with $m_1 = m_2 = m$, and with $q = m$, $|e(t)| \leqslant C' h^{m+1}$, $t \in I_\varepsilon$. Consider then (3.19): for each $n \geqslant r$, $e_n(t_n + \tau h)$ is given by the representation (3.13b) [where $|y^{(m+1)}(z)| \leqslant M < \infty$ for $z \in I \setminus I_\varepsilon$]. Furthermore, by observing that (3.19) may now be viewed as a perturbation of (3.6) with known perturbation term $\{\Psi_r(t_n + c_i h) - \hat{\Psi}_r(t_n + c_i h)\}$ and by then applying the arguments mentioned before, we are led to the result of Theorem 2.

4. Starting Values: An Alternative Method

The methods investigated in the previous section are all based on a direct discretization of (1.1); the singular behavior of the derivatives of the exact solution y at $t = 0$ was reflected in the particular choice of the projecting spaces. Since, as has already been pointed out [5, 9], it is the order of the starting errors that will eventually govern the accuracy of the numerical approximations on I, the computation of approximate values near $t = 0$ can be based on a suitable discretization of the system (2.1) [whose solution $z(t) := (v(t), w(t))^{\mathrm{T}} \in \mathbf{R}^2$ is smooth on I] rather than of (1.1).

Let

$$\bar{S}_m^{(-1)}(Z_N) := \left\{ \bar{u} : \bar{u}|_{\sigma_n} = \bar{u}_n(t) := \sum_{j=0}^{m} a_{n,j}\tau^j,\ \tau = (t - t_n)/h,\right.$$

$$\left. a_{n,j} \in \mathbf{R}^2\ (n = 0, \ldots, N-1) \right\} \tag{4.1}$$

and set $\bar{u}(t) := (u^{\mathrm{I}}(t), u^{\mathrm{II}}(t))^{\mathrm{T}}$. If we introduce the set

$$X_n := \{t_n + c_i h : 0 \leqslant c_0 < \cdots < c_m = 1\}$$

$(n = 0, \ldots, N-1)$, then the requirement that the element $\bar{u} \in \bar{S}_m^{(-1)}(Z_N)$ satisfies the system (2.1) on $X(N) := \bigcup_{n=0}^{N-1} X_n$ leads to

$$u^{\mathrm{I}}(t_n + c_i h) = h \int_0^{c_i} \frac{(n+\tau)^\alpha}{(c_i - \tau)^\alpha} G(t_n + c_i h, t_n + \tau h)$$
$$\times \gamma_1(t_n + \tau h, u_n^{\mathrm{I}}(t_n + \tau h), u_n^{\mathrm{II}}(t_n + \tau h))\, d\tau + \hat{F}_{n,1}(t_n + c_i h) \tag{4.2a}$$

and

$$u_n^{\mathrm{II}}(t_n + c_i h) = \frac{h^{1-2\alpha}}{(n + c_i)^\alpha} \int_0^{c_i} \frac{1}{(c_i - \tau)^\alpha} G(t_n + c_i h, t_n + \tau h)$$
$$\times \gamma_2(t_n + \tau h, u_n^{\mathrm{I}}(t_n + \tau h), u_n^{\mathrm{II}}(t_n + \tau h))\, d\tau$$
$$+ \hat{F}_{n,2}(t_n + c_i h) \qquad (i = 0, \ldots, m;\ n = 0, \ldots, N-1), \tag{4.2b}$$

with $\hat{F}_{n,l}$ defined by analogy to (3.5).

Under the hypotheses given in Section 2, (4.2) defines a unique element $\bar{u} \in \bar{S}_m^{(-1)}(Z_N)$ for all sufficiently small $h > 0$. Furthermore, by recalling that $\alpha = \frac{1}{2}$, it can be shown that

$$\|z(t) - \bar{u}(t)\|_\infty \leqslant Ch^{m+1} \qquad \text{for all} \quad t \in I; \tag{4.3}$$

hence,

$$|v(t) - u^{\mathrm{I}}(t)| \leqslant C_1 h^{m+1}, \quad |w(t) - u^{\mathrm{II}}(t)| \leqslant C_2 h^{m+1}, \qquad t \in I, \tag{4.4}$$

and thus, by (2.4),

$$\begin{aligned} |y(t) - [u^{\mathrm{I}}(t) + t^\beta u^{\mathrm{II}}(t)]| &\leqslant |v(t) - u^{\mathrm{I}}(t)| + t^\beta |w(t) - u^{\mathrm{II}}(t)| \\ &\leqslant (C_1 + C_2 T^\beta) h^{m+1}, \qquad t \in I. \end{aligned}$$

We conclude with some observations. First, if we multiply (4.2b) by $(t_n + c_i h)^\alpha$ and then add the new equation to (4.2a), we arrive at

$$\begin{aligned} &u_n^{\mathrm{I}}(t_n + c_i h) + (t_n + c_i h)^\beta u_n^{\mathrm{II}}(t_n + c_i h) \\ &\quad = h^{1-\alpha} \int_0^{c_i} (c_i - \tau)^{-\alpha} G(t_n + c_i h, t_n + \tau h) \gamma\big(t_n + \tau h, u_n^{\mathrm{I}}(t_n + \tau h) \\ &\qquad + (t_n + \tau h)^\beta u_n^{\mathrm{II}}(t_n + \tau h)\big)\, d\tau + \hat{F}_{n,1}(t_n + c_i h) \\ &\qquad + (t_n + c_i h)^\beta \hat{F}_{n,2}(t_n + c_i h) \qquad (i = 0, \ldots, m); \end{aligned} \tag{4.5}$$

this is, of course, identical with the collocation equation (3.4) where the roles of u_n^{I} and u_n^{II} are being taken by p_n and q_n, provided $m_1 = m_2 = m$ [see (2.7)]. The second remark relates to a remark made after (2.4): due to the possibility of the system (2.1) being unstable, the procedure (4.2) based on this system will only be used during the "transient phase" (nonsmoothness) of the exact solution to furnish starting values of suitable accuracy. Note also that both the method suggested by de Hoog and Weiss [5] [based on the system (1.2) there] and above method (4.2) [for the system (2.1)] cannot be extended to include equations whose weakly singular kernels are characterized by $\alpha \neq \frac{1}{2}$, owing to the presence of the singularity $t^{-\alpha}$ in (2.1b). On the other hand, the direct method (3.4) is not subject to this restriction for the values of α.

5. Runge–Kutta Methods for (1.1)

We touch briefly on a particular aspect of the discretization of the collocation equation (3.4): If the polynomials $\{l_j(\tau): j = 0, \ldots, \mu - 1\}$ [with $\mu = m_1 + m_2 + 1$; see (2.7)] denote the Lagrange fundamental polynomials with respect to the μ (distinct) parameters $\{c_i: i = 0, \ldots, \mu - 1\}$, and if we define

$$a_{ij}(\alpha) := \int_0^{c_i} (c_i - \tau)^{-\alpha} l_j(\tau)\, d\tau \qquad (i = 0,\ldots,\mu; j = 0,\ldots,\mu - 1),$$

then the evaluation of the integrals on the right-hand side of (3.4) by (weighted) interpolatory quadrature based on the μ abscissas $\{c_i : i = 0,\ldots,\mu - 1\}$ leads to

$$Y_i^{(n)} = h^{1-\alpha} \sum_{j=0}^{\mu-1} a_{ij}(\alpha) G(t_n + c_i h, t_n + c_j h) \gamma(t_n + c_j h, Y_j^{(n)}) + \tilde{F}_n(t_n + c_i h) \qquad (i = 0,\ldots,\mu) \tag{5.1a}$$

and

$$y_{n+1} = Y_\mu^{(n)} \qquad (n = 0,\ldots,N-1); \tag{5.1b}$$

here $Y_i^{(n)}$ denotes an approximation to $u_n(t_n + c_i h)$ [and hence to $y(t_n + c_i h)$; recall that $c_\mu = 1$], and $\tilde{F}_n(t)$ is a suitably accurate discretization of (3.5). Note that (5.1a) requires certain values of $G(t,s)$ for $s > t$ that have to be defined.

According to the theory on the attainable degree of precision in (weighted) interpolatory quadrature [observe that the number of abscissas, μ, satisfies $\mu \geqslant \min(m_1 + 1, m_2 + 1) = q + 1$], a glimpse at the appropriately modified error equation (3.6) (where the integrals have been replaced by the above interpolatory quadrature formulas and the corresponding expression for the quadrature errors) shows that the statement (3.10) of Theorem 1 remains essentially valid; i.e., we obtain

$$|y(t_n) - y_n)| \leqslant Ch^{q+1} \qquad [n = 1,\ldots,N-1], \tag{5.2}$$

provided, of course, that the quadrature formulas used in $\tilde{F}_n(t_n + c_i h)$ have a consistent degree of precision.

The scheme (5.1) represents an implicit Runge–Kutta method of the Pouzet type for (1.1) whose order is given by $p = q + 1 \geqslant 1$ $(0 < \alpha < 1)$. If (5.1) is replaced by

$$Y_i^{(n)} = h^{1-\alpha} \sum_{j=0}^{\mu-1} a_{ij}(\alpha) G(t_n + d_j h, t_n + c_j h) \gamma(t_n + c_j h, Y_j^{(n)}) + \tilde{F}_n(t_n + c_i h) \qquad (i = 0,\ldots,\mu), \tag{5.3a}$$

$$y_{n+1} = Y_\mu^{(n)} \qquad (n = 0,\ldots,N-1), \tag{5.3b}$$

then we have an implicit Runge–Kutta method of Bel'tyukov type; the parameters $\{d_j\}$ are to be chosen such that $d_j \geqslant c_j$, $j = 0,\ldots,\mu - 1$.

At present, very little is known about the attainable order of (explicit or implicit) Runge–Kutta methods for Volterra integral equations with weakly singular kernels, especially for methods of Bel'tyukov type. A study of these equations is the subject of a recent thesis [1].

6. Numerical Illustrations

Consider the simple test equation (mentioned already in Section 2)

$$y(t)=1-\int_0^t (t-s)^{-1/2} y(s)\,ds, \qquad t\in I:=[0,1], \tag{6.1}$$

with solution $y(t)=\exp(\pi t)\,\mathrm{erfc}(\pi^{1/2}t^{1/2})$ [where $\mathrm{erfc}(u)=2\pi^{-1/2}\int_u^\infty \exp(-t^2)\,dt$]. In the representation (2.4) of y we have

$$v(t)=\exp(\pi t) \qquad \text{and} \qquad w(t)=t^{-1/2}\exp(\pi t)\left[\mathrm{ercf}(\pi^{1/2}t^{1/2})-1\right].$$

We also have $y(t)=1-2t^{1/2}+O(t)$, as $t\to 0_+$.

In order to illustrate the theoretical convergence results given in Section 3, we have solved (6.1) numerically (i) with the collocation scheme (3.4), and (ii) with the modified collocation scheme based on the representation (3.12) of the approximate solution $u(t)$.

i. We have solved (3.4) for $\mu=m_1+m_2+1=3$, with $(m_1,m_2)=(0,2)$, $(1,1)$ and $(2,0)$, respectively, for $h=\frac{1}{5}$, $\frac{1}{10}$, and $\frac{1}{20}$, and for the following three choices (A,B,C) of the collocation points c_0, c_1, c_2, and c_3:

	c_0	c_1	c_2	c_3
A	0.033732053372	0.282593677396	0.746460414456	1
B	0.099194170728	0.450131500784	0.835289713103	1
C	0.25	0.5	0.75	1

In A, the c_i are the abscissas of weighted interpolatory quadrature with weight function $(1-\tau)^{-1/2}$, exact for the integrands $\tau^{i/2}$, $i=0,\ldots,6$. In B, the c_i are the corresponding abscissas for integrands τ^i, $i=0,\ldots,6$ (for both choices, compare te Riele's Table 2.1 [9]). The moment integrals in (3.4) and (3.5) were evaluated analytically.

In Table 1 we give for each of the 27 experiments [i.e., for three different (m_1,m_2), for three different h, and for three different sets of c_i] the number of correct digits in the computed approximate solution at the point $t=1.0$ and the minimum number of correct digits on the whole integration interval $[0,1]$. In all cases this minimum was attained in $t=h$. The results confirm the theoretical convergence results of Section 3. Moreover, the choice of the collocation points c_i appears to be of crucial importance for the accuracy. We conclude that the most accurate scheme is the one with $m_1=m_2=1$, with choice A for the collocation points $c_0,\ldots,c_3$. The results obtained with

l. Collocation Scheme (3.4) Applied to (6.1): Number of Correct Digits

	$(m_1, m_2) = (0,2)$		(1,1)		(2,0)	
$h^{-1} = N$	in 1.0	minimum on [0.1]	in 1.0	minimum on [0,1]	in 1.0	minimum on [0,1]
5	5.6	4.5	6.6	5.5	6.3	5.2
10	6.3	5.1	7.4	6.2	7.1	5.9
20	7.0	5.7	8.2	7.0	7.8	6.5
5	5.0	4.2	5.6	4.8	5.4	4.6
10	5.6	4.5	6.3	5.3	6.0	5.0
20	6.1	4.8	7.1	5.8	6.8	5.5
5	4.4	3.6	5.0	4.2	4.8	4.0
10	4.9	3.9	5.7	4.7	5.4	4.4
20	5.5	4.2	6.5	5.2	6.1	4.8

Modified Collocation Scheme Applied to (6.1): Number of Correct Digits

, $\tilde{c}_i$	$h^{-1} = N$	Correct in 1.0	Minimum correct[a]	Correct in $t = \varepsilon$, in $t = \varepsilon + h$
, $\tilde{A}$	5	3.8	3.2 (0.4)	5.5, 3.2
	10	4.5	3.7 (0.3)	6.5, 3.7
	20	5.1	4.4 (0.25)	7.3, 4.4
, $\tilde{B}$	5	5.3	3.6 (0.4)	4.8, 3.6
	10	5.0	4.2 (0.3)	5.5, 4.2
	20	5.4	4.9 (0.25)	6.3, 4.9
', $\tilde{C}$	5	3.8	3.6 (0.6)	4.2, 3.7
	10	4.3	4.0 (0.4)	4.9, 4.1
	20	4.8	4.4 (0.35)	5.7, 4.7
, $\tilde{A}$	5	3.9	3.6 (0.6)	6.1, 3.6
	10	4.6	4.2 (0.5)	6.9, 4.2
	20	5.2	4.9 (0.5)	7.7, 4.9
', $\tilde{B}$	5	4.6	4.0 (0.6)	5.1, 4.0
	10	5.9	4.6 (0.5)	5.9, 4.6
	20	5.8	5.4 (0.45)	6.6, 5.4
', $\tilde{C}$	5	4.0	4.0 (0.8)	4.5, 4.2
	10	4.5	4.4 (0.7)	5.2, 4.6
	20	5.0	4.9 (0.65)	6.0, 5.2

at which minimum is obtained is in parentheses.

this scheme are practically the same as the results obtained with te Riele's scheme A [9, Table 4.2, Example 6, scheme A, $m = 3$, $r = 2$].

ii. The modified collocation scheme discussed after (3.12) was solved for $\varepsilon = 0.2$ (resp. $\varepsilon = 0.4$), $m = 1$ ($\mu_0 = 3$, $\mu_1 = 1$). For $t \leqslant \varepsilon$ the collocation points were chosen as above (A, B, and C) so that for $t \leqslant \varepsilon$ this scheme coincides with the above one for $(m_1, m_2) = (1, 1)$. For $\varepsilon < t \leqslant T\,(= 1.0)$ the collocation points A, B, and C were replaced by the following points $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$, respectively:

	$\tilde{c}_0$	$\tilde{c}_1$
$\tilde{A}$	0.306101188813	1.0
$\tilde{B}$	0.4	1.0
$\tilde{C}$	0.5	1.0

As with A and B the $\tilde{c}_0$ and $\tilde{c}_1$ in $\tilde{A}$ and $\tilde{B}$ are the abscissas of weighted interpolatory quadrature with weight function $(1-\tau)^{-1/2}$, exact for the integrands $\tau^{i/2}$ (resp. τ^i), $i = 0, 1, 2$ [9, Table 2.1]. The moment integrals occurring in the modified collocation scheme were evaluated analytically.

In Table 2 we present results similar to those in Table 1. Moreover, we give the number of correct digits in $t = \varepsilon$ and in $t = \varepsilon + h$, in order to illustrate the loss of accuracy caused by the transition from the relatively expensive four-point collocation scheme (applied for $t \leqslant \varepsilon$) to the relatively cheap two-point scheme (for $t > \varepsilon$). The value of t where the minimum number of correct digits is attained is indicated in parentheses.

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